PLURI-CANONICAL MAPS OF VARIETIES OF MAXIMAL ALBANESE DIMENSION IN POSITIVE CHARACTERISTIC

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Abstract. We show that if X is a nonsingular projective variety of general type over an algebraically closed field k of positive characteristic and X has maximal Albanese dimension and the Albanese map is separable, then $|4K_X|$ induces a birational map.

1. Introduction

Let X be a nonsingular projective variety of general type over an algebraically closed field k, $n = \dim X$ and K_X be a canonical divisor. Since K_X is big, for any sufficiently large positive integer m, the linear series $|mK_X|$ induces a birational map. It is an important problem to bound this integer m. For char $k = 0$, by a result of [HM06], [Takayama06], [Tsuji06] and [Tsuji07], there exists a non-effective bound for m which only depends on the dimension n. When X has maximal Albanese dimension, [CH02] and [JLT11] show the optimal result that $|3K_X|$ is birational. Furthermore, [CH02] shows that if the Albanese dimension is $n-1$, then $|6K_X|$ is birational, and if the Albanese dimension is $n-2$, then $|7K_X|$ is birational.

In this paper, we will generalize these results to positive characteristic.

Theorem 1.1. *(See Theorem 5.5) Let X be a smooth projective variety of general type over an algebraically closed field k of characteristic $p > 0$. If X has maximal Albanese dimension and the Albanese map is separable, then $|4K_X|$ induces a birational map.*

Our strategy is similar to that in [CH02] where the Fourier-Mukai transform (Lemma 3.1) is used repeatedly to produce sections of mK_X. Their approach uses multiplier ideals and Kawamata-Viehweg vanishing in an essential way. In positive characteristic, the theory of Fourier-Mukai transforms still applies, and multiplier ideals can be replaced by test ideals. However, Kawamata-Viehweg vanishing is known to fail. Inspired by [Hacon11], [Mustaţă11] and [Schwede11], we replace Kawamata-Viehweg vanishing by the Frobenius map and Serre vanishing. Combining this with the Fourier-Mukai transform, we obtain that $|4K_X|$ is birational. It seems that new ideas are required to investigate the third pluri-canonical map.

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2. Asymptotic Test Ideals

Suppose that X is a smooth n-dimensional variety over an algebraically closed field k of characteristic $p > 0$. Let ω_X denote the canonical line bundle on X. We denote $F : X \to X$ the absolute Frobenius morphism, that is given by the identity on the topological space, and by taking the p-th power on regular functions. Let $\mathrm{Tr} : F_*\omega_X \to \omega_X$ be the trace map and $\mathrm{Tr}^e : F^e_*\omega_X \to \omega_X$ be the e-th iteration of the trace map.

We follow the definitions given in [Mustaţă11]. For other equivalent definitions, see [BMS08] and [Schwede11]. Given a nonzero ideal $\mathfrak{a}$ in $\mathcal{O}_X$, the image $\mathrm{Tr}^e(\mathfrak{a}\cdot\omega_X)$ can be written as $\mathfrak{a}^{[1/p^e]} \cdot \omega_X$ for some ideal $\mathfrak{a}^{[1/p^e]}$ in $\mathcal{O}_X$. Given a positive real number λ, one can show that

$$\left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} \subseteq \left(\mathfrak{a}^{\lceil \lambda p^{e+1} \rceil}\right)^{[1/p^{e+1}]}$$

for every $e \geqslant 1$ where $\lceil t \rceil$ means the smallest integer $\geqslant t$. Hence, there is an ideal $\tau(\mathfrak{a}^\lambda)$, called the **test ideal** of $\mathfrak{a}$ of exponent λ, that is equal to $\left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]}$ for all e large enough.

Test ideals have many similar properties to multiplier ideals. If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\tau(\mathfrak{a}^\lambda) \subseteq \tau(\mathfrak{b}^\lambda)$ for all $\lambda \geqslant 0$. If m is a positive integer, then $\tau(\mathfrak{a}^{m\lambda}) = \tau((\mathfrak{a}^m)^\lambda)$.

One can also define an asymptotic version of test ideals similar to asymptotic multiplier ideals. Suppose that $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X ($\mathfrak{a}_m \cdot \mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$) and λ is a positive real number. If m and l are two positive integers such that $\mathfrak{a}_m$ is nonzero, then

$$\tau(\mathfrak{a}_m^{\lambda/m}) = \tau((\mathfrak{a}_m^l)^{\lambda/ml}) \subseteq \tau(\mathfrak{a}_{ml}^{\lambda/ml}).$$

By the Noetherian property, there is a unique ideal $\tau(\mathfrak{a}_\bullet^\lambda)$, called the **asymptotic test ideal** of $\mathfrak{a}_\bullet$ of exponent λ, such that $\tau(\mathfrak{a}_\bullet^\lambda) = \tau(\mathfrak{a}_m^{\lambda/m})$ for all m large enough and sufficiently divisible.

For linear series, let D be a Cartier divisor on X such that $h^0(X, \mathcal{O}_X(mD)) \neq 0$ for some positive integer m. We then define $\tau(\lambda \cdot \|D\|) = \tau(\mathfrak{a}_\bullet^\lambda)$ where $\mathfrak{a}_m$ is the base ideal of the linear series $|mD|$. Then by definition, $\tau(\lambda/r \cdot \|rD\|) = \tau(\lambda \cdot \|D\|)$ for every positive integer r. If D is a $\mathbb{Q}$-divisor such that $h^0(X, \mathcal{O}_X(mD)) \neq 0$ for some positive integer m satisfying that mD is Cartier, then we put $\tau(\lambda \cdot \|D\|) = \tau(\lambda/r \cdot \|rD\|)$ for some $r > 0$ such that rD is Cartier.

3. Fourier-Mukai Transform

We recall some facts about the Fourier-Mukai Transform from [Mukai81]. Let A be an abelian variety of dimension g, $\hat{A}$ be the dual abelian variety and P be the normalized Poincaré line bundle on $A \times \hat{A}$. For any $\hat{a} \in \hat{A}$, we let $P_{\hat{a}} = P|_{A\times\hat{a}}$. The **Fourier-Mukai functor** $R\hat{S} : D(A) \to D(\hat{A})$ is given by $R\hat{S}(\mathcal{F}) = Rp_{\hat{A},*}(p_A^*\mathcal{F} \otimes P)$. There is a corresponding functor $RS : D(\hat{A}) \to D(A)$ such that $RS \circ R\hat{S} = (-1_A)^*[-g]$ and $R\hat{S} \circ RS = (-1_{\hat{A}})^*[-g]$.

We will need the following result.

Proposition 3.1. *Let $\mathcal{F}$ be a non-zero coherent sheaf on A such that $H^i(A, \mathcal{F} \otimes P_{\hat{a}}) = 0$ for all $\hat{a} \in \hat{A}$ and all $i > 0$. If $\mathcal{F} \to k(a)$ is a surjective morphism for some $a \in A$, then the induced map $H^0(A, \mathcal{F} \otimes P_{\hat{a}}) \to k(a)$ is surjective for general $\hat{a} \in \hat{A}$.*

Proof. See [Hacon11, 2.1]. □

4. Vanishing Theorems

Suppose $f : X \to A$ is a nontrivial morphism where X is a smooth variety of general type over an algebraic closed field k of characteristic $p > 0$ and A is an abelian variety. Let K_X be a canonical divisor. Since K_X is big, we have $K_X \sim_{\mathbb{Q}} H + E$ where H is an ample $\mathbb{Q}$-divisor and E is an effective $\mathbb{Q}$-divisor. Let $\Delta = (1-\epsilon)K_X + \epsilon E$, where $\epsilon \in \mathbb{Q}$ and $0 < \epsilon < 1$. Fix a positive integer l such that $l\Delta$ is Cartier. Although Δ is not necessarily effective, since K_X is big and E is effective, we have the Iitaka dimension $\kappa(X, l\Delta) \geqslant 0$. For any positive integer r, let $\mathcal{F}_r = \mathcal{O}_X((r+1)K_X) \otimes \tau(\|r\Delta\|)$.

Let $\mathfrak{a}_{\mathfrak{m}}$ be the base ideal of the linear series $|ml\Delta|$. By the definition of the asymptotic test ideal, we can fix a positive integer m' sufficiently large and divisible such that $\tau(\|r\Delta\|) = \tau(\mathfrak{a}_\bullet^{r/l}) = \tau(\mathfrak{a}_{m'}^{r/m'l})$. We may assume $m' = rm$ for some positive integer m. Then $\tau(\|r\Delta\|) = \tau(\mathfrak{a}_{rm}^{1/ml})$. For every $e \gg 0$, we have $\tau(\mathfrak{a}_{rm}^{1/ml}) = \left(\mathfrak{a}_{rm}^{\lceil p^e/ml \rceil}\right)^{[1/p^e]}$. Hence, the iterated trace map Tr^e gives a surjection

$$\mathrm{Tr}^e : F_*^e(\mathfrak{a}_{rm}^{\lceil p^e/ml \rceil} \cdot \mathcal{O}_X(K_X)) \to \tau(\|r\Delta\|) \cdot \mathcal{O}_X(K_X).$$

Tensoring with $\mathcal{O}_X(rK_X)$, we have a surjection

$$F_*^e(\mathfrak{a}_{rm}^{\lceil p^e/ml \rceil} \cdot \mathcal{O}_X((rp^e+1)K_X)) \to \mathcal{F}_r.$$

Let $\widetilde{\mathcal{F}_{r,e}} = \mathfrak{a}_{rm}^{\lceil p^e/ml \rceil} \cdot \mathcal{O}_X((rp^e+1)K_X)$. Then the surjection above is $F_*^e\widetilde{\mathcal{F}_{r,e}} \to \mathcal{F}_r$. Since $\mathfrak{a}_{rm}$ is the base ideal of $|rml\Delta|$, the evaluation gives a surjection

$$H^0(X, \mathcal{O}_X(rml\Delta)) \otimes \mathcal{O}_X(-rml\Delta) \to \mathfrak{a}_{rm},$$

hence a surjection

$$V_{r,e} \otimes \mathcal{O}_X(-rml\lceil p^e/ml \rceil \Delta) \to \mathfrak{a}_{rm}^{\lceil p^e/ml \rceil},$$

where $V_{r,e} = \mathrm{Sym}^{\lceil p^e/ml \rceil} H^0(X, \mathcal{O}_X(rml\Delta))$. Tensoring with $\mathcal{O}_X((rp^e+1)K_X)$, we have a surjection

$$\mathcal{F}_{r,e} = V_{r,e} \otimes \mathcal{O}_X(-rml\lceil p^e/ml \rceil \Delta + (rp^e+1)K_X) \to \widetilde{\mathcal{F}_{r,e}},$$

hence a surjection $F_*^e\mathcal{F}_{r,e} \to \mathcal{F}_r$ since F^e is affine.

Lemma 4.1. *Fix $r > 0$. Then $R^if_*(F_*^e\mathcal{F}_{r,e}) = 0$ for all $i > 0$ and all e large enough.*

Proof. First, we prove that $R^if_*\mathcal{F}_{r,e} = 0$ for all $i > 0$ and all e large enough. Since $V_{r,e}$ is a vector space over k, we only need to show that

$$R^if_*\mathcal{O}_X(-rml\lceil p^e/ml \rceil \Delta + (rp^e+1)K_X) = 0.$$

But

$$\begin{aligned} & -rml\lceil p^e/ml \rceil \Delta + (rp^e+1)K_X \\ = & -rmls\Delta + (rmls - rt + 1)K_X \\ = & \ (1-rt)K_X + rmls(K_X - \Delta), \end{aligned}$$

where $p^e = mls - t$ and $0 \leqslant t < ml$. Noticing that $K_X - \Delta \sim_{\mathbb{Q}} \epsilon H$ which is ample, we may apply Serre vanishing. For each value of $t \in [0, ml-1]$, we have $R^if_*\mathcal{O}_X((1-rt)K_X + rmls(K_X - \Delta)) = 0$ for all s large enough, i.e., all e large enough. Thus $R^if_*\mathcal{F}_{r,e} = 0$.

Now, since F^e is exact and commutes with f, we have

$$R^if_*(F^e_*\mathcal{F}_{r,e}) = R^i(f\circ F^e)_*\mathcal{F}_{r,e} = R^i(F^e\circ f)_*\mathcal{F}_{r,e} = R^iF^e_*(f_*\mathcal{F}_{r,e}) = 0$$

for all $i > 0$ and all e large enough. $\square$

Lemma 4.2. *Fix $r > 0$. There is an integer $M > 0$ such that $H^i(A, f_*(F^e_*\mathcal{F}_{r,e})\otimes P) = 0$ for all $i > 0$, $e > M$ and $P \in Pic^0(A)$.*

Proof. The proof is similar to that of Lemma 4.1. By Lemma 4.1 and the projection formula, $R^if_*(F^e_*\mathcal{F}_{r,e}\otimes f^*P) = 0$ for all $i > 0$ and e large enough. Hence, by a spectral sequence argument, it suffices to prove that $H^i(X, F^e_*\mathcal{F}_{r,e}\otimes f^*P) = 0$ or equivalently that $H^i(X, \mathcal{F}_{r,e}\otimes f^*P^{\otimes p^e}) = 0$. We only need to show that

$$H^i(X, \mathcal{O}_X(-rml\lceil p^e/ml\rceil\Delta + (rp^e+1)K_X)\otimes f^*P^{\otimes p^e}) = 0.$$

Assume that $p^e = mls - t$ where $0 \leqslant t < ml$. Since $K_X - \Delta \sim_{\mathbb{Q}} \epsilon H$ is ample, by Fujita vanishing, for each value of t, there is an $M_t > 0$ such that for all $e > M_t$ and all nef line bundles $\mathcal{N}$ on X, we have

$$H^i(X, \mathcal{O}_X((1-rt)K_X + rmls(K_X-\Delta))\otimes\mathcal{N}) = 0.$$

Let $M = \max\{M_t\}$, then

$$H^i(X, \mathcal{O}_X(-rml\lceil p^e/ml\rceil\Delta + (rp^e+1)K_X)\otimes\mathcal{N}) = 0$$

for all $e > M$ and all nef line bundles $\mathcal{N}$. In particular, we can take $\mathcal{N} = f^*P^{\otimes p^e}$. The lemma follows. $\square$

5. Main Results

Fix a positive integer m such that $\tau(\|r\Delta\|) = \tau(\mathfrak{a}_{rm}^{1/ml})$. Let $\mathcal{I}$ be an ideal sheaf in $\mathcal{O}_X$. In our applications, $\mathcal{I} = \mathcal{O}_X$ or $\mathcal{I} = \mathcal{I}_x$, where $\mathcal{I}_x$ is the maximal ideal of closed point. The composition of the two surjections, $F^e_*\mathcal{F}_{r,e} \to \mathcal{F}_r$ and $\mathcal{F}_r \to \mathcal{F}_r\otimes\mathcal{O}_X/\mathcal{I}$, is still surjective. We define $(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}}$ to be the kernel of this composition. Then $(F^e_*\mathcal{F}_{r,e})_{\mathcal{O}_X} = F^e_*\mathcal{F}_{r,e}$. Since the composition $(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to F^e_*\mathcal{F}_{r,e} \to \mathcal{F}_r \to \mathcal{F}_r\otimes\mathcal{O}_X/\mathcal{I}$ is 0, it factors through the kernel of $\mathcal{F}_r \to \mathcal{F}_r\otimes\mathcal{O}_X/\mathcal{I}$, which is $\mathcal{F}_r\otimes\mathcal{I}$ providing that,

$$\text{the intersection of the co-supports of } \tau(\|r\Delta\|) \text{ and } \mathcal{I} \text{ is empty.} \qquad (*)_r$$

We have a map $(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to \mathcal{F}_r\otimes\mathcal{I}$, and by the 5-lemma, it is surjective. This is summarized in the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & (F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \arrow[r] \arrow[d] & F^e_*\mathcal{F}_{r,e} \arrow[r] \arrow[d] & \mathcal{F}_r\otimes\mathcal{O}_X/\mathcal{I} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F}_r\otimes\mathcal{I} \arrow[r] & \mathcal{F}_r \arrow[r] & \mathcal{F}_r\otimes\mathcal{O}_X/\mathcal{I} \arrow[r] & 0
\end{tikzcd}$$

Remark 5.1. The condition $(*)_r$ is true if $\mathcal{I} = \mathcal{O}_X$ or $\mathcal{I} = \mathcal{I}_x$ where x is not in the co-support of $\tau(\|r\Delta\|)$. And $(*)_r$ implies $(*)_s$ if $r \geqslant s$, since $\tau(\|r\Delta\|) \subseteq \tau(\|s\Delta\|)$.

Suppose x is a point in X such that x is not in the co-support of the ideals $\tau(\|r\Delta\|)$ or $\mathcal{I}$. Then the restriction to the point x gives a surjection $\mathcal{F}_r\otimes\mathcal{I} \to \mathcal{F}_r\otimes k(x) \cong k(x)$. Hence, a surjection $\phi_{r,e,\mathcal{I}} : (F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to \mathcal{F}_r\otimes k(x)$. Let $(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x}$ be the kernel of $\phi_{r,e,\mathcal{I}}$.

Let $f : X \to A$ be a nontrivial separable morphism where A is an abelian variety.

Theorem 5.2. *Fix $e > 0$ and r a positive integer. Let $\mathcal{I}$ be an ideal sheaf in $\mathcal{O}_X$ satisfying $(*)_r$. Suppose that x is a point in X such that*

1. *x is not in the co-support of $\tau(\|r\Delta\|)$ or $\mathcal{I}$,*
2. *$f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \neq f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}}$,*
3. *$H^i(A, f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \otimes P) = 0$ for all $i > 0$ and all $P \in Pic^0(A)$.*

Then the homomorphism $H^0(X, (F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \otimes f^*P) \to H^0(X, \mathcal{F}_r \otimes \mathcal{I} \otimes f^*P \otimes k(x)) \cong k(x)$ induced by $\phi_{r,e,\mathcal{I}}$ is surjective for general $P \in Pic^0(A)$. Moreover, x is not a base point of $\mathcal{F}_r \otimes \mathcal{I} \otimes f^*P$ for general $P \in Pic^0(A)$.

Proof. Pushing forward the exact sequence

$$0 \to (F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to (F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to \mathcal{F}_r \otimes k(x) \to 0,$$

we have

$$0 \to f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to f_*(\mathcal{F}_r \otimes k(x)) \to R^1f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to \cdots .$$

Let $a = f(x)$. Since f is separable, we have that a is reduced, hence $f_*(\mathcal{F}_r \otimes k(x)) \cong k(a)$. By assumption, $f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}}$ is not an isomorphism, which implies that the kernel of $k(a) \to R^1f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x}$ is not 0. But the kernel is a sub-sheaf of $k(a)$ who has no non-zero sub-sheaf other than itself. Hence the kernel is $k(a)$ and we have an exact sequence

$$0 \to f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to k(a) \to 0.$$

Applying Proposition 3.1 to the surjection $f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to k(a)$, we have the surjection $H^0(f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \otimes P) \to k(a)$ for general $P \in \mathrm{Pic}^0(A)$. Hence, the theorem follows. For the moreover part, noticing that the surjection factors through $H^0(\mathcal{F}_r \otimes \mathcal{I} \otimes f^*P)$, we have the induced homomorphism $H^0(\mathcal{F}_r \otimes \mathcal{I} \otimes f^*P) \to k(x)$ is also surjective. $\square$

The following corollary is useful in the case of maximal Albanese dimension.

Corollary 5.3. *Suppose f is finite over an open subset U in A. Fix $e > 0$ and r a positive integer. Let $\mathcal{I}$ be an ideal sheaf in $\mathcal{O}_X$ satisfying $(*)_r$. Suppose that x is a point in X such that*

1. *x is not in the co-support of $\tau(\|r\Delta\|)$ or $\mathcal{I}$,*
2. *$a = f(x) \in U$,*
3. *$H^i(A, f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \otimes P) = 0$ for all $i > 0$ and all $P \in Pic^0(A)$.*

Then the conclusion of Theorem 5.2 still holds.

Proof. By Theorem 5.2, we only need to show that $f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \neq f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}}$. Recall that we have an exact sequence

$$0 \to f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to f_*(\mathcal{F}_r \otimes k(x)) \to R^1f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to \cdots .$$

If $f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} = f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}}$, we have that the map $f_*(\mathcal{F}_r \otimes k(x)) \to R^1f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x}$ is nonzero. So the stalk of $R^1f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x}$ at a is nonzero. But as f is finite over a, the higher direct images are 0 at a, a contradiction. $\square$

Theorem 5.2 also gives information on the base locus of $\mathcal{O}_X(2K_X) \otimes f^*P$ for general $P \in \mathrm{Pic}^0(A)$.

Corollary 5.4. *Fix $e > M$ as in Lemma 4.2. Suppose that x is a point in X such that*

1. *x is not in the co-support of $\tau(\|\Delta\|)$,*

(2) $f_*(F^e_*\mathcal{F}_{1,e})_{\mathcal{O}_X,x} \neq f_*(F^e_*\mathcal{F}_{1,e})$.

Then x is not a base point of $\mathcal{F}_1 \otimes f^*P$ for general $P \in Pic^0(A)$. Hence x is not a base point of $\mathcal{O}_X(2K_X) \otimes f^*P$ for general $P \in Pic^0(A)$.

Proof. The first part of the corollary follows directly from Theorem 5.2 and Lemma 4.2. And the second part follows from the facts that $\mathcal{F}_1 = \mathcal{O}_X(2K_X) \otimes \tau(\|\Delta\|)$ and $\tau(\|\Delta\|)$ is an ideal. □

We are ready to prove the main result.

Theorem 5.5. *Let X be a smooth projective variety of general type over an algebraic closed field k of characteristic $p > 0$. If X has maximal Albanese dimension and the Albanese map is separable, then $|4K_X|$ induces a birational map.*

Proof. Suppose A is the Albanese variety and $f : X \to A$ is the Albanese map. Since f is generically finite, there is an open subset U of A such that f is finite over U. And since f is separable, we have that $K_X = f^*K_A + R_f = R_f \geqslant 0$. As usual, we let $\mathcal{F}_r = \mathcal{O}_X((r+1)K_X) \otimes \tau(\|r\Delta\|)$ and $\mathfrak{a}_m = \mathfrak{bs}(|ml\Delta|)$. We fix a positive integer m such that $\tau(\|r\Delta\|) = \tau(\mathfrak{a}_{rm}^{1/ml})$ for $r = 1, 2, 3$ and define

$$\widetilde{\mathcal{F}_{r,e}} = \mathfrak{a}_{rm}^{\lceil p^e/ml\rceil} \cdot \mathcal{O}_X((rp^e+1)K_X),$$

$$\mathcal{F}_{r,e} = V_{r,e} \otimes \mathcal{O}_X(-rml\lceil p^e/ml\rceil\Delta + (rp^e+1)K_X),$$

where $V_{r,e} = \mathrm{Sym}^{\lceil p^e/ml\rceil} H^0(X, \mathcal{O}_X(rml\Delta))$. Let

$$\widetilde{\mathcal{F}_{1,e}}^{-} = \mathfrak{a}_m^{\lceil p^e/ml\rceil} \cdot \mathcal{O}_X(p^eK_X) = \widetilde{\mathcal{F}_{1,e}} \otimes \mathcal{O}_X(-K_X),$$

$$\mathcal{F}^-_{1,e} = V_{1,e} \otimes \mathcal{O}_X(-ml\lceil p^e/ml\rceil\Delta + p^eK_X) = \mathcal{F}_{1,e} \otimes \mathcal{O}_X(-K_X).$$

Claim: For $e \gg 0$, we have $R^if_*(F^e_*\mathcal{F}^-_{1,e}) = 0$ and $H^i(A, f_*(F^e_*\mathcal{F}^-_{1,e}) \otimes P) = 0$ for all $i > 0$ and $P \in \mathrm{Pic}^0(A)$.

The claim follows immediately from the proofs of Lemma 4.1 and 4.2.

We fix an integer $e \gg 0$ such that $H^i(A, f_*\mathcal{G} \otimes P) = 0$ holds for all $i > 0$ and all $\mathcal{G} \in \{\mathcal{F}_{1,e}, \mathcal{F}_{2,e}, \mathcal{F}_{3,e}, \mathcal{F}^-_{1,e}\}$. By a general point $a \in A$, we mean a point $a \in U$. By a general point $x \in X$, we mean a point $x \in f^{-1}(U)$ such that x is not in the co-supports of $\mathfrak{a}_m^{\lceil p^e/ml\rceil}$ or $\tau(\|3\Delta\|)$ (hence, not in the co-supports of $\tau(\|2\Delta\|)$ or $\tau(\|\Delta\|)$ by Remark 5.1) and that x is not in the support of K_X.

Our strategy is: First, by Theorem 5.2, we have that x is not a base point of $\mathcal{F}_1 \otimes f^*P$ for general $P \in \mathrm{Pic}^0(A)$. Then, by comparing $\mathcal{F}_1$ and $\mathcal{F}_2$ via $\mathcal{F}^-_{1,e}$, we show that x is not a base point of $\mathcal{F}_2 \otimes f^*P$ for all $P \in \mathrm{Pic}^0(A)$. Using this fact, we show that $\mathcal{F}_2 \otimes f^*P$ separates points for general $P \in \mathrm{Pic}^0(A)$. Finally, by comparing $\mathcal{F}_2$ and $\mathcal{F}_3$ via $\mathcal{F}^-_{1,e}$, we have that $\mathcal{F}_3 \otimes f^*P$ separates points for all $P \in \mathrm{Pic}^0(A)$. Hence, so does $\mathcal{F}_3$.

Step 1. By Lemma 4.2 and Corollary 5.3 with $r = 1$ and $\mathcal{I} = \mathcal{O}_X$, we have that for general $x \in X$, the homomorphism

$$H^0(X, F^e_*\mathcal{F}_{1,e} \otimes f^*P) \to H^0(X, \mathcal{F}_1 \otimes f^*P \otimes k(x)) \cong k(x)$$

is surjective for general $P \in \mathrm{Pic}^0(A)$.

Step 2. We show that for general $x \in X$, the homomorphism

$$H^0(X, F^e_*\mathcal{F}_{2,e} \otimes f^*Q) \to H^0(X, \mathcal{F}_2 \otimes f^*Q \otimes k(x)) \cong k(x)$$

is surjective for all $Q \in \mathrm{Pic}^0(A)$.

Since $\mathfrak{a}_m^{\lceil p^e/ml \rceil}$ is an ideal, we have an inclusion $\widetilde{\mathcal{F}_{1,e}}^{-} \to \mathcal{O}_X(p^e K_X)$. Tensoring with the vector bundle $\mathcal{F}_{1,e}$, we have an inclusion

$$\widetilde{\mathcal{F}_{1,e}}^{-} \otimes \mathcal{F}_{1,e} \to \mathcal{O}_X(p^e K_X) \otimes \mathcal{F}_{1,e},$$

whose cokernel is supported on the co-support of $\mathfrak{a}_m^{\lceil p^e/ml \rceil}$. Pushing forward by the Frobenius, we get another inclusion

$$F^e_*(\widetilde{\mathcal{F}_{1,e}}^{-} \otimes \mathcal{F}_{1,e}) \to F^e_*(\mathcal{O}_X(p^e K_X) \otimes \mathcal{F}_{1,e}) \cong \mathcal{O}_X(K_X) \otimes F^e_*(\mathcal{F}_{1,e}),$$

whose cokernel is still supported on the co-support of $\mathfrak{a}_m^{\lceil p^e/ml \rceil}$. Hence, the induced morphism

$$\alpha : F^e_*(\widetilde{\mathcal{F}_{1,e}}^{-} \otimes \mathcal{F}_{1,e}) \otimes k(x) \to \mathcal{O}_X(K_X) \otimes F^e_*(\mathcal{F}_{1,e}) \otimes k(x)$$

is an isomorphism providing that x is general. Since $\mathfrak{a}_m^2 \subseteq \mathfrak{a}_{2m}$, we have a morphism $\widetilde{\mathcal{F}_{1,e}}^{-} \otimes \widetilde{\mathcal{F}_{1,e}} \to \widetilde{\mathcal{F}_{2,e}}$. Combining with the morphism $\mathcal{F}_{1,e} \to \widetilde{\mathcal{F}_{1,e}}$, we have that

$$\widetilde{\mathcal{F}_{1,e}}^{-} \otimes \mathcal{F}_{1,e} \to \widetilde{\mathcal{F}_{1,e}}^{-} \otimes \widetilde{\mathcal{F}_{1,e}} \to \widetilde{\mathcal{F}_{2,e}}.$$

On the other hand, since $\tau(\|\Delta\|)$ and $\tau(\|2\Delta\|)$ are ideals, the induced inclusions $\mathcal{F}_1 \otimes k(x) \to \mathcal{O}_X(2K_X) \otimes k(x)$ and $\mathcal{F}_2 \otimes k(x) \to \mathcal{O}_X(3K_X) \otimes k(x)$ are both isomorphisms providing that x is general. Hence, there is a morphism

$$\mathcal{F}_1 \otimes k(x) \to \mathcal{O}_X(K_X) \otimes \mathcal{F}_1 \otimes k(x) \cong \mathcal{O}_X(3K_X) \otimes k(x) \cong \mathcal{F}_2 \otimes k(x).$$

Combining the discussion above and the trace maps $F^e_*\mathcal{F}_{r,e} \to F^e_*\widetilde{\mathcal{F}_{r,e}} \to \mathcal{F}_r$, we have the following commutative diagram:

$$\begin{tikzcd}
F^e_*\mathcal{F}_{1,e} \otimes k(x) \arrow[r] \arrow[d, two heads] & \mathcal{O}_X(K_X) \otimes F^e_*\mathcal{F}_{1,e} \otimes k(x) \arrow[r, "\simeq", "\alpha^{-1}"'] \arrow[d, two heads] & F^e_*(\widetilde{\mathcal{F}_{1,e}}^{-} \otimes \mathcal{F}_{1,e}) \otimes k(x) \arrow[r] \arrow[d, two heads] & F^e_*\widetilde{\mathcal{F}_{2,e}} \otimes k(x) \arrow[d, two heads] \\
\mathcal{F}_1 \otimes k(x) \arrow[r, "\simeq"] & \mathcal{O}_X(K_X) \otimes \mathcal{F}_1 \otimes k(x) \arrow[r, equal] & \mathcal{O}_X(K_X) \otimes \mathcal{F}_1 \otimes k(x) \arrow[r, "\simeq"] & \mathcal{F}_2 \otimes k(x).
\end{tikzcd}$$

The surjectivities of the first, second and last vertical maps are induced by the surjectivity of $F^e_*\mathcal{F}_{r,e} \to \mathcal{F}_r$. The third map is the same as the second map.

Noticing that $\mathcal{F}_{1,e}$ is a vector bundle, we have that the morphism $\widetilde{\mathcal{F}_{1,e}}^{-} \otimes \mathcal{F}_{1,e} \to \widetilde{\mathcal{F}_{2,e}}$ is equivalent to a morphism $\widetilde{\mathcal{F}_{1,e}}^{-} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_{1,e}, \widetilde{\mathcal{F}_{2,e}})$. We have

$$\begin{aligned}
F^e_*\widetilde{\mathcal{F}_{1,e}}^{-} &\to F^e_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_{1,e}, \widetilde{\mathcal{F}_{2,e}}) \\
&\to \mathcal{H}om_{\mathcal{O}_X}(F^e_*\mathcal{F}_{1,e}, F^e_*\widetilde{\mathcal{F}_{2,e}}) \\
&\to \mathcal{H}om_{\mathcal{O}_X}(F^e_*\mathcal{F}_{1,e} \otimes k(x), F^e_*\widetilde{\mathcal{F}_{2,e}} \otimes k(x))
\end{aligned}$$

which is the top row of the commutative diagram above. This induces a morphism between $\mathcal{F}_1 \otimes k(x)$ and $\mathcal{F}_2 \otimes k(x)$. Indeed, for any $a \in \mathcal{F}_1 \otimes k(x)$, we have some $b \in F^e_*\mathcal{F}_{1,e} \otimes k(x)$ (maybe not unique) mapped to a by the first vertical arrow in the commutative diagram. Applying the top row induced by $F^e_*\widetilde{\mathcal{F}_{1,e}}^{-}$ and then the

last vertical arrow on b, we get some $c \in \mathcal{F}_2 \otimes k(x)$ which is independent of the choice of b since the diagram commutes. Hence, we have a morphism

$$F^e_* \widetilde{\mathcal{F}_{1,e}}^- \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1 \otimes k(x), \mathcal{F}_2 \otimes k(x)) \cong k(x).$$

This morphism is nonzero, and hence surjective, providing that x is not in the support of the effective divisor K_X, since the bottom row of the commutative diagram are all isomorphisms in this case. Combining with the surjection $\mathcal{F}^-_{1,e} \to \widetilde{\mathcal{F}_{1,e}}^-$, we have the following surjection:

$$F^e_* \mathcal{F}^-_{1,e} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1 \otimes k(x), \mathcal{F}_2 \otimes k(x)) \cong k(x).$$

For any $Q \in \mathrm{Pic}^0(A)$, we can pick $P \in \mathrm{Pic}^0(A)$ such that P and $Q \otimes P^\vee$ are both general. Applying Corollary 5.3 to the surjection $F^e_* \mathcal{F}^-_{1,e} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1 \otimes k(x), \mathcal{F}_2 \otimes k(x))$, since $Q \otimes P^\vee$ is general, we get a surjection

$$H^0(X, F^e_* \mathcal{F}^-_{1,e} \otimes f^*(Q \otimes P^\vee)) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_1 \otimes f^*P \otimes k(x), \mathcal{F}_2 \otimes f^*Q \otimes k(x)).$$

Combining with the fact from Step 1, that $H^0(X, F^e_* \mathcal{F}_{1,e} \otimes f^*P) \to H^0(X, \mathcal{F}_1 \otimes f^*P \otimes k(x))$ is surjective, we have a surjection

$$H^0(X, F^e_* \mathcal{F}^-_{1,e} \otimes F^e_* \mathcal{F}_{1,e} \otimes f^*Q) \to H^0(X, \mathcal{F}_2 \otimes f^*Q \otimes k(x)).$$

Noticing that by the commutative diagram above, this surjection factors through $H^0(X, F^e_* \mathcal{F}_{2,e} \otimes f^*Q)$. Therefore, we have that the homomorphism

$$H^0(X, F^e_* \mathcal{F}_{2,e} \otimes f^*Q) \to H^0(X, \mathcal{F}_2 \otimes f^*Q \otimes k(x))$$

is surjective for all $Q \in \mathrm{Pic}^0(A)$.

Step 3. We show that for general $x_1, x_2 \in X$ and general $P \in \mathrm{Pic}^0(A)$, we can find a section in $F^e_* \mathcal{F}_{2,e} \otimes f^*P$ which induces a section in $\mathcal{F}_2 \otimes f^*P$ vanishing at x_1 but not x_2.

We only need to show that the map

$$H^0(X, (F^e_* \mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \otimes f^*P) \to H^0(X, \mathcal{F}_2 \otimes \mathcal{I}_{x_1} \otimes f^*P \otimes k(x_2)) \cong k(x_2)$$

is surjective. Noticing that x_1 is not in the co-support of $\tau(\|2\Delta\|)$, we have that $\mathcal{I}_{x_1}$ satisfies $(*)_2$. Applying Corollary 5.3 with $r = 2$ and $\mathcal{I} = \mathcal{I}_{x_1}$, it suffices to check $H^i(A, f_*(F^e_* \mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \otimes P) = 0$ for all $i > 0$ and all $P \in \mathrm{Pic}^0(A)$.

First we show that $R^i f_*(F^e_* \mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} = 0$ for $i > 0$. Pushing forward the exact sequence

$$0 \to (F^e_* \mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \to F^e_* \mathcal{F}_{2,e} \to \mathcal{F}_2 \otimes k(x_1) \to 0$$

gives

$$f_*(F^e_* \mathcal{F}_{2,e}) \to k(a_1) \to R^1 f_*(F^e_* \mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \to R^1 f_*(F^e_* \mathcal{F}_{2,e}) = 0$$

and

$$R^i f_*(F^e_* \mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \cong R^i f_*(F^e_* \mathcal{F}_{2,e}) = 0$$

for all $i \geqslant 2$, where $a_1 = f(x_1)$. As in the proof of Theorem 5.2 (notice that $(F^e_* \mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} = (F^e_* \mathcal{F}_{2,e})_{\mathcal{O}_X, x_1}$), one sees that $f_*(F^e_* \mathcal{F}_{2,e}) \to k(a_1)$ is surjective, so $R^1 f_*(F^e_* \mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} = 0$.

Now, by a spectral sequence argument, we only need to show that $H^i(X, (F^e_* \mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \otimes f^*P) = 0$ for all $i > 0$ and all $P \in \mathrm{Pic}^0(A)$. We have the short exact sequence

$$0 \to (F^e_* \mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \otimes f^*P \to F^e_* \mathcal{F}_{2,e} \otimes f^*P \to \mathcal{F}_2 \otimes f^*P \otimes k(x_1) \to 0.$$

By taking the cohomology, we have

$$H^0(X, F^e_*\mathcal{F}_{2,e}\otimes f^*P) \to k(x_1) \to H^1(X, (F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}}\otimes f^*P) \to H^1(X, F^e_*\mathcal{F}_{2,e}\otimes f^*P) = 0$$

and

$$H^i(X, (F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \otimes f^*P) \cong H^i(X, F^e_*\mathcal{F}_{2,e} \otimes f^*P) = 0,$$

for all $i \geqslant 2$. Since by Step 2, $H^0(X, F^e_*\mathcal{F}_{2,e} \otimes f^*P) \to k(x_1)$ is surjective, we have $H^1(X, (F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \otimes f^*P) = 0$.

Step 4. We show that for general $x_1, x_2 \in X$ and all $Q \in \mathrm{Pic}^0(A)$, we can find a section in $F^e_*\mathcal{F}_{3,e} \otimes f^*Q$ which induces a section in $\mathcal{F}_3 \otimes f^*Q$ vanishing at x_1 but not x_2.

For any general points x_1 and x_2 and any $Q \in \mathrm{Pic}^0(A)$, we may pick $P \in \mathrm{Pic}^0(A)$ such that P and $Q \otimes P^\vee$ are both general. Similar to Step 2, for $i = 1$ or 2, we have the following commutative diagram:

$$\begin{tikzcd}
F^e_*\mathcal{F}_{2,e}\otimes k(x_i) \arrow[r] \arrow[d] & \mathcal{O}_X(K_X)\otimes F^e_*\mathcal{F}_{2,e}\otimes k(x_i) \arrow[r, "\simeq"] \arrow[d] & F^e_*(\widetilde{\mathcal{F}_{1,e}}^{-}\otimes \mathcal{F}_{2,e})\otimes k(x_i) \arrow[r] \arrow[d] & F^e_*\widetilde{\mathcal{F}_{3,e}}\otimes k(x_i) \arrow[d] \\
\mathcal{F}_2\otimes k(x_i) \arrow[r, "\simeq"] & \mathcal{O}_X(K_X)\otimes \mathcal{F}_2\otimes k(x_i) \arrow[r, equal] & \mathcal{O}_X(K_X)\otimes \mathcal{F}_2\otimes k(x_i) \arrow[r, "\simeq"] & \mathcal{F}_3\otimes k(x_i).
\end{tikzcd}$$

And we have that

$$F^e_*\mathcal{F}^-_{1,e} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2 \otimes k(x_i), \mathcal{F}_3 \otimes k(x_i)) \cong k(x_i)$$

is surjective. We may apply Corollary 5.3 and get that

$$H^0(X, F^e_*\mathcal{F}^-_{1,e} \otimes f^*(Q \otimes P^\vee)) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_2 \otimes f^*P \otimes k(x_2), \mathcal{F}_3 \otimes f^*Q \otimes k(x_2))$$

is surjective.

By Step 3, we have a section $s \in H^0(X, F^e_*\mathcal{F}_{2,e} \otimes f^*P)$ restricting to 0 in $\mathcal{F}_2 \otimes f^*P \otimes k(x_1)$ and to nonzero in $\mathcal{F}_2 \otimes f^*P \otimes k(x_2)$. And by the discussion above, we have a section $s^- \in H^0(X, F^e_*\mathcal{F}^-_{1,e}\otimes f^*(Q\otimes P^\vee))$ inducing a nonzero homomorphism between $\mathcal{F}_2 \otimes f^*P \otimes k(x_2)$ and $\mathcal{F}_3 \otimes f^*Q \otimes k(x_2)$. Hence, $s^- \otimes s$ gives a section in $H^0(X, F^e_*\mathcal{F}_{3,e} \otimes f^*Q)$ restricting to 0 in $\mathcal{F}_3 \otimes f^*Q \otimes k(x_1)$ and to nonzero in $\mathcal{F}_3 \otimes f^*Q \otimes k(x_2)$.

Step 5. By Step 4, for all $Q \in \mathrm{Pic}^0(A)$, we have a surjection

$$H^0(X, F^e_*\mathcal{F}_{3,e} \otimes f^*Q) \to H^0(X, \mathcal{F}_3 \otimes f^*Q \otimes k(x_1, x_2)),$$

where $k(x_1, x_2)$ is the skyscraper sheaf supported on $\{x_1, x_2\}$. Since this surjection factors through $H^0(X, \mathcal{F}_3 \otimes f^*Q)$, we have that $\mathcal{F}_3 \otimes f^*Q$ separates general points for all $Q \in \mathrm{Pic}^0(A)$.

Since $\mathcal{F}_3 = \mathcal{O}_X(4K_X) \otimes \tau(\|3\Delta\|)$ and $\tau(\|3\Delta\|)$ is an ideal, we can conclude that $|4K_X|$ induces a birational map.

$\square$

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